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SOME APPROXIMATION PROPERTIES OF MODIFIED SZÁSZ-MIRAKJAN-BASKAKOV OPERATORS

PRASHANTKUMAR PATEL* AND VISHNU NARAYAN MISHRA

ABSTRACT. The generalization of well known Szász-Mirakjan operators was introduction by *G C Jain, Approximation of functions by a new class of linear operators, Journal of the Australian Mathematical Society, 13(3):271–276, 1972*. In *P Patel and V N Mishra, Jain-Baskakov operators and its different generalization, Acta Mathematica Vietnamica, 40:715–733, 2015* introduced the integral modification of Jain operators and discussed its different generalization. In this manuscript, we extend the study of the operators introduced by Patel & Mishra and discussed some direct results in ordinary approximation for this operators.

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1. INTRODUCTION

Using the generalized poisson distribution the following positive linear operators was introduced by Jain [5]: for $f \in C([0,\ \infty))$; $\beta \in [0,1)$

$$J_n^{[\beta]}(f,x) = \sum_{k=0}^{\infty} \omega_\beta(k,nx) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$\omega_\beta(k,nx) = nx\,(nx+k\beta)^{k-1}\,\frac{e^{-(nx+k\beta)}}{k!};\ \text{for } k=0,1,2,\ldots;x\in[0,\infty). \tag{2}$$

In the particular case $\beta = 0$, $J_0^{[\beta]}$, $n \in \mathbb{N}$, turn into classical Szász-Mirakjan operators [6, 12]. We mention that, a Kantorovich-type extension of these sequence of operators was discussed by Umar and Razi [14]. Very recently, integral modification of these operators having a weight function of some Beta basis functions was introduced in [8, 13]. The Voronosvkaja-type asymptotic formula of the operators (1) was presented by Farcaş [2], the modification of these asymptotic formula was discussed in [1].

*corresponding Author

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Furthermore, the integral modification of these operators having a weight function of some Baskakov basis function defined for $f \in C([0,\infty))$ as

$$(3)\qquad K_n^{[\beta]}(f,x) = (n-1)\sum_{k=1}^{\infty}\omega_\beta(k,nx)\int_0^\infty p_{n,k-1}(t)f(t)dt + e^{-nx}f(0),$$

where

$$p_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}$$

and $\omega(k,nx)$ as defined in (2). The above integral operators and its different generalizations were discussed by Patel and Mishra [9]. In the same paper both the authors also introduced the following generalization of the operators $K_n^{[\beta]}$ with parameter $c>0$ as:

$$(4)\qquad K_{n,c}^{[\beta]}(f,x) = \frac{(n-c)}{c}\sum_{k=1}^{\infty}\omega_\beta(k,nx)\int_0^\infty p_{n,k-1,c}(t)f(t)dt + e^{-nx}f(0),$$

where

$$p_{n,k-1,c}(t) = c\frac{\Gamma(\frac{n}{c}+k-1)}{\Gamma(k)\Gamma(\frac{n}{c})}\frac{(ct)^{k-1}}{(1+ct)^{\frac{n}{c}+k-1}}.$$

As a special case, i.e., $c=1$, the operators (4) reduce to the Jain-Baskakov operators defined by (3). These operators have different approximation properties than the operators (3). For the particular case $c=1$ and $\beta=0$, the modified Jain-Baskakov operators (4) equal to the classical Szász-Mirakjan-Baskakov operators discussed in [11, 3]. Some researchers contributed in the direction of generalizing the operators with parameter $c>0$ can be found in [8, 10, 4, 7]. In the next section, we prove a weighted Korovkin theorem by obtaining rate of approximation in terms of modulus of continuity. In section 3, we give a Voronovskaya type asymptotic formula. For the convergence of these operators, the condition $\beta\to 0$ is needed. Thus throughout this paper, we take $\beta := \beta_n = \frac{1}{n}$ $(n\in\mathbb{N})$ and $C([0,\infty))$ denote by the set of continuous functions on interval $[0,\infty)$.

Lemma 1.1. [5, 2] *For the first few moments of the operators $J_n^{[\frac{1}{n}]}(f,x)$, we have*

(1) $J_n^{[\frac{1}{n}]}(1,x) = 1,$

(2) $J_n^{[\frac{1}{n}]}(t,x) = \dfrac{nx}{n-1},$

(3) $J_n^{[\frac{1}{n}]}(t^2,x) = \dfrac{(nx)^2}{(n-1)^2} + \dfrac{n^2x}{(n-1)^3},$

(4) $J_n^{[\frac{1}{n}]}(t^3,x) = \dfrac{(nx)^3}{(n-1)^3} + \dfrac{3n^3x^2}{(n-1)^4} + \dfrac{x(n^3+2n^2)}{(n-1)^5},$

(5) $J_n^{[\frac{1}{n}]}(t^4,x) = \dfrac{(nx)^4}{(n-1)^4} + \dfrac{6n^4x^3}{(n-1)^5} + \dfrac{x^2(7n^4+8n^3)}{(n-1)^6} + \dfrac{x\left(n^4+8n^3+6n^2\right)}{(n-1)^7}.$

Lemma 1.2. *For the moments of the operators $K_{n,c}^{[\frac{1}{n}]}(f,x)$, we have*

(1) $K_{n,c}^{[\frac{1}{n}]}(1,x) = 1;$

(2) $K_{n,c}^{[\frac{1}{n}]}(t,x) = \dfrac{n^2x}{(n-1)(n-2c)}$, *for $n>2c$;*

(3) $K_{n,c}^{[\frac{1}{n}]}(t^2,x) = \frac{n^2}{(n-2c)(n-3c)}\left\{\frac{\left(1-2n+2n^2\right)x}{(n-1)^3}+\frac{n^2x^2}{(n-1)^2}\right\}$, *for* $n>3c$;

(4) $K_{n,c}^{[\frac{1}{n}]}(t^3,x) = \frac{n^2}{(n-2c)(n-3c)(n-4c)}\left\{\frac{n^4x^3}{(n-1)^3}+\frac{3n^2x^2\left(1-2n+2n^2\right)}{(n-1)^4}\right.$ $\left.+\frac{x\left(2-8n+15n^2-12n^3+6n^4\right)}{(n-1)^5}\right\}$, *for* $n>4c$;

(5) $K_{n,c}^{[\frac{1}{n}]}(t^4,x) = \frac{n^2}{(n-2c)(n-3c)(n-4c)(n-5c)}\left\{\frac{n^6x^4}{(n-1)^4}+\frac{6n^4\left(1-2n+2n^2\right)x^3}{(n-1)^5}\right.$ $+\frac{n^2\left(11-44n+84n^2-72n^3+36n^4\right)x^2}{(n-1)^6}$ $\left.+\frac{\left(6-36n+101n^2-152n^3+144n^4-72n^5+24n^6\right)x}{(n-1)^7}\right\}$, *for* $n>5c$.

The proof follows from Lemma 1.1 and the linearity of the operator.

Lemma 1.3. *For the central moments of the operators* $K_{n,c}^{[\frac{1}{n}]}(f,x)$, *we have*

(1) $K_{n,c}^{[\frac{1}{n}]}(t-x,x) = \left\{\frac{n^2}{(n-1)(n-2c)}-1\right\}x$;

(2) $K_{n,c}^{[\frac{1}{n}]}((t-x)^2,x) = \frac{\left(6c^2-c(5+12c)n+\left(1+4c+6c^2\right)n^2+cn^3\right)x^2}{(n-1)^2(n-3c)(n-2c)}+\frac{\left(n^2-2n^3+2n^4\right)x}{(n-1)^3(n-3c)(n-2c)}$, *for* $n>3c$;

(3) $K_{n,c}^{[\frac{1}{n}]}((t-x)^3,x) = x^3\left\{\frac{-24c^3+2c^2(13+36c)n-3c\left(3+14c+24c^2\right)n^2}{(n-1)^3(n-4c)(n-3c)(n-2c)}\right.$ $\left.+\frac{\left(1+6c+6c^2+24c^3\right)n^3+c(3+10c)n^4}{(n-1)^3(n-4c)(n-3c)(n-2c)}\right\}+\frac{3n^2(4c(n-1)+n)(1-2n+2n^2)x^2}{(n-1)^4(n-4c)(n-3c)(n-2c)}$ $+\frac{n^2(2-8n+15n^2-12n^3+6n^4)x}{(n-1)^5(n-4c)(n-3c)(n-2c)}$, *for* $n>4c$;

(4) $K_{n,c}^{[\frac{1}{n}]}((t-x)^4,x) = x^4\left\{\frac{120c^4-2c^3(77+240c)n+c^2(71+376c+720c^2)n^2}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^4}\right.$ $+\frac{-2c(7+48c+102c^2+240c^3)n^3+(1+8c-18c^2-104c^3+120c^4)n^4}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^4}$ $\left.+\frac{2c(3+20c+43c^2)n^5+3c^2n^6}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^4}\right\}$ $+\frac{6n^2x^3(1-2n+2n^2)(20c^2-c(9+40c)n+(1+8c+20c^2)n^2+cn^3)}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^5}$ $+n^2x^2\left\{\frac{-40c+8(1+25c)n-(29+460c)n^2+12(4+45c)n^3}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^6}\right.$ $\left.+\frac{-24(1+15c)n^4+120cn^5+12n^6}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^6}\right\}$ $+\frac{n^2x(6-36n+101n^2-152n^3+144n^4-72n^5+24n^6)}{(n-2c)(n-3c)(n-4c)(n-5c)(n-1)^7}$, *for* $n>5c$.

The proof follows from Lemma 1.2 and the linearity of the operator.

2. Rate of Convergence

Let $B_{x^2}([0,\infty)) = \{f : [0,\infty) \to \mathbb{R} : |f(x)| \leq K_f(1+x^2), \text{ for all } x \in [0,\infty)\}$, where K_f is a positive constant depending only on f. Let $C_{x^2}([0,\infty)) = B_{x^2}([0,\infty)) \cap C([0,\infty))$. In this section, we establish the rate of convergence in the space
$C^*_{x^2}([0,\infty)) = \left\{ f \in C_{x^2}([0,\infty)) : \lim\limits_{x\to\infty} \frac{|f(x)|}{1+x^2} \text{ is finite} \right\}$. The space $C^*_{x^2}([0,\infty))$ is norm linear space with norm $\|f\|_{x^2} = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$.
Let $A > 0$. The usual modulus of continuity of f on the closed interval $[0, A]$ is defined by

$$\omega_A(f,\delta) = \sup\{|f(x) - f(t)| : |x-t| \leq \delta,\ x,t \in [0,A]\}.$$

It is well known that, for function $f \in C^*_{x^2}([0,\infty))$, $\lim\limits_{\delta\to 0} \omega_A(f,\delta) = 0$. The next theorem gives the rate of convergence of the operators $K^{[\frac{1}{n}]}_{n,c}(f,\cdot)$ to f, for all $f \in C^*_{x^2}([0,\infty))$:

Theorem 2.1. *Let $f \in C^*_{x^2}([0,\infty))$ and let $\omega_{A+1}(f,\delta), (A > 0)$ be its modulus of continuity on the finite interval $[0, A+1] \subset [0,\infty)$. Then,*

$$\|K^{[\frac{1}{n}]}_{n,c}(f,\cdot) - f\|_{C([0,A])} \leq N_f(1+A^2)\delta_n^2 + 2\omega_{A+1}(f,\delta_n), \tag{5}$$

where $\delta_n = \left[\left| \frac{6c^2 - c(5+12c)n + (1+4c+6c^2)n^2 + cn^3}{(n-1)^2(n-3c)(n-2c)} \right| A^2 + \left| \frac{n^2 - 2n^3 + 2n^4}{(n-1)^3(n-3c)(n-2c)} \right| A \right]^{\frac{1}{2}}$
and N_f is a positive constant depending on f.

Proof. Let $x \in [0, A]$ and $t \leq A + 1$. It is clear that

$$|f(t) - f(x)| \leq \omega_{A+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{A+1}(f,\delta), \tag{6}$$

where $\delta > 0$. On the other hand, for $x \in [0, A]$ and $t \geq A+1$, using the fact that $t - x \geq 1$, we have

$$|f(t)-f(x)| \leq K_f(1+x^2+t^2) \leq K_f(2+3x^2+2(t-x)^2) \leq N_f(1+A^2)(t-x)^2, \tag{7}$$

where $N_f = 6K_f$. From inequalities (6) and (7), we get for all $x \in [0, A]$ and $t \geq 0$ that

$$|f(t) - f(x)| \leq N_f(1+A^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{A+1}(f,\delta),$$

and therefore by linear properties of the operators $K^{[\frac{1}{n}]}_{n,c}$, we obtain

$$|K^{[\frac{1}{n}]}_{n,c}(f,x) - f(x)| \leq N_f(1+A^2)K^{[\frac{1}{n}]}_{n,c}((t-x)^2,x) + \left(1 + \frac{1}{\delta}K^{[\frac{1}{n}]}_{n,c}(|t-x|,x)\right)\omega_{A+1}(f,\delta).$$

By Cauchy-Schwarz inequality, we have

$$|K^{[\frac{1}{n}]}_{n,c}(f,x) - f(x)| \leq N_f(1+A^2)K^{[\frac{1}{n}]}_{n,c}((t-x)^2,x) + \left(1 + \frac{1}{\delta}\left[K^{[\frac{1}{n}]}_{n,c}((t-x)^2,x)\right]^2\right)\omega_{A+1}(f,\delta),$$

Using Lemma 1.3 and then taking supremum over the interval $[0, A]$ on both sides of the inequality (5) is archive. Hence, the proof is completed. □

Corollary 2.2. *For all $f \in C^*_{x^2}([0,\infty))$, the sequence of operators $\left\{K_{n,c}^{[\frac{1}{n}]}(f,\cdot)\right\}$ converges uniformly to f on $[0, A]$ $(A > 0)$.*

3. A Voronovskaya-Type Asymptotic formula

In this section, we prove a Voronovskaya-type asymptotic formula for the operators $K_{n,c}^{[\frac{1}{n}]}$ given by (4). We first need the following lemma.

Lemma 3.1. $\lim_{n\to\infty} n^2 K_{n,c}^{[\frac{1}{n}]}((t-x)^4, x) = 3x^2(2+cx)^2$, *uniformly with respect to $x \in [0, A]$ with $A > 0$.*

Proof. The proof follows from Lemma 1.3. □

Theorem 3.2. *For every $f \in C^*_{x^2}([0,\infty))$ such that $f', f'' \in C^*_{x^2}([0,\infty))$,*

$$\lim_{n\to\infty} n\left(K_{n,c}^{[\frac{1}{n}]}(f,x) - f(x)\right) = x(1+2c)f'(x) + \frac{x}{2}(2+cx)f''(x)$$

uniformly for all $x \in [0, A]$.

Proof. We note that f, f' and f'' are in $C^*_{x^2}([0,\infty))$. Define

$$\Omega(t,x) = \begin{cases} \frac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2 f''(x)}{(t-x)^2} & \text{if } t \neq x \\ 0, & \text{if } t = x. \end{cases}$$

One can notice that, $\Omega(f,x) = 0$ and $\Omega(f,x) \in C^*_{x^2}([0,\infty))$. By Taylor's theorem

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!} f''(x) + (t-x)^2\Omega(f,x).$$

Applying the operators $K_{n,c}^{[\frac{1}{n}]}$ both sides of the above equality, we obtain

$$\begin{aligned} n\left(K_{n,c}^{[\frac{1}{n}]}(f,x) - f(x)\right) &= nf'(x)K_{n,c}^{[\frac{1}{n}]}(t-x,x) + n\frac{f''(x)}{2}K_{n,c}^{[\frac{1}{n}]}((t-x)^2,x) \\ &\quad + nK_{n,c}^{[\frac{1}{n}]}((t-x)^2\Omega(t,x),x). \end{aligned} \tag{8}$$

By the Cauchy-Schwarz inequality, we get for the second term on the right-hand side of (8) that

$$n\left|K_{n,c}^{[\frac{1}{n}]}((t-x)^2\Omega(t,x),x)\right| \le \left(n^2 K_{n,c}^{[\frac{1}{n}]}((t-x)^4,x)\right)^{\frac{1}{2}} \left(K_{n,c}^{[\frac{1}{n}]}(\Omega^2(t,x),x)\right)^{\frac{1}{2}}. \tag{9}$$

Now, observe that $\Omega(x,x) = 0$ and $\Omega(\cdot,x) \in C^*_{x^2}([0,\infty))$. Therefore it follows from Corollary 2.2 that

$$\lim_{n\to\infty} K_{n,c}^{[\frac{1}{n}]}(\Omega^2(t,x),x) = \Omega^2(x,x) = 0$$

uniformly for all $x \in [0, A]$, $(A > 0)$. Now, by Lemma 3.1, we see that

$$\lim_{n\to\infty} nK_{n,c}^{[\frac{1}{n}]}((t-x)^2\Omega(t,x),x) = 0. \tag{10}$$

On the other hand, from Lemma 1.3, one can observe that

$$\lim_{n\to\infty} nK_{n,c}^{[\frac{1}{n}]}((t-x),x) = x(1+2c) \tag{11}$$

and

$$\lim_{n\to\infty} nK_{n,c}^{[\frac{1}{n}]}((t-x)^2,x) = x(2+cx) \tag{12}$$

uniformly for all $x \in [0,A]$. Then taking the limit as $n \to \infty$ in (8) and using (9), (10), (11) and (12), we have

$$\lim_{n\to\infty} n\left(K_{n,c}^{[\frac{1}{n}]}(f,x) - f(x)\right) = x(1+2c)f'(x) + \frac{x}{2}(2+cx)f''(x),$$

uniformly for all $x \in [0,A]$. Hence, the proof is completed. □

Remark: In particular, if $c = 0$, then the operators $K_{n,c}^{[\frac{1}{n}]}(f,\cdot)$, reduce to the Jain-Baskakov operators recently introduced in [9]. We obtain the following conclusion of the above asymptotic formula for the Jain-Baskakov operator in the ordinary approximation as follows: for f, f' and $f'' \in C_{x^2}^*([0,\infty))$,

$$\lim_{n\to\infty} n\left(K_{n,0}^{[\frac{1}{n}]}(f,x) - f(x)\right) = x\left(f'(x) + f''(x)\right)$$

uniformly for all $x \in [0,A]$ $(A > 0)$.

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Department of Mathematics, St. Xaviers College, Ahmedabad-380 009 (Gujarat), India
Sardar Vallabhbhai National Institute of Technology, Surat-395 007 (Gujarat), India
E-mail address: prashant225@gmail.com

Sardar Vallabhbhai National Institute of Technology, Surat-395 007 (Gujarat), India
Opposite-Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad-224 001 (Uttar Pradesh), India
E-mail address: vishnu_narayanmishra@yahoo.co.in; vishnunarayanmishra@gmail.com